1. Heavy Quark and Soft Collinear Effective Theory

Revised August 2011 by C. W. Bauer (LBNL) and M. Neubert (U. Mainz).

1.1. Effective Field theories

Quantum field theories represent the most precise computational tool for describing physics at the highest energies. One of their characteristic features is that they almost inevitably involve multiple length scales. When trying to determine the value of an observable, quantum field theory demands that all possible virtual states and hence all particles and states be included in the calculation. The different particles have widely different masses, hence the principal sensitivity of the final prediction to many scales. This fact represents a formidable challenge from a practical point of view. No realistic quantum field theories can be solved exactly, so that one has to resort to approximation schemes; these, however, are typically able to provide a reliable description only for a single scale at a time.

Effective field theories provide a general theoretical framework to deal with the multi-scale problems of realistic quantum field theories. This framework acknowledges that only a single scale at a time can be handled well; simultaneously, however, it provides an organizational scheme whereby the other scales are not omitted but allowed to play their role in a separate step of the computation. The philosophy and basic principles of this approach are very generic, and correspondingly effective field theories represent the modern method of choice in practically all areas of high-energy physics, from the low energy scales of atomic and nuclear physics to the high energy scales of (partly yet unknown) particle physics. Effective field theories can play a role both within analytic perturbative computations, as well as in the context of non-perturbative numerical simulations. For some early references on effective field theories, see [1,2,3,4].

One of the simplest applications of effective theories to particle physics is to describe an underlying theory that is only probed at energy scales $E < \Lambda$. Any particle with mass $m > \Lambda$ cannot be produced as a real state and therefore only leads to short-distance virtual effects. Thus, one can construct an effective theory in which the quantum fluctuations of such heavy particles are "integrated out" from the generating functional integral for Green functions. This results in a simpler theory containing only those degrees of freedom that are relevant to the energy scales under consideration. In fact, the standard model of particle physics itself is an effective theory of some yet unknown, more fundamental theory.

The development of any effective theory starts by identifying the relevant degrees of freedom that are relevant to describe the physics at a given length (or energy) scale, and constructing the Lagrangian describing the interactions among these fields. Short-distance quantum fluctuations associated with much smaller length scales are absorbed into the coefficients of the various operators in the effective Lagrangian. These coefficients are determined in a matching procedure, by requiring that the effective theory reproduces the matrix elements of the full theory up to power corrections. In many cases the effective Lagrangian exhibits enhanced symmetries compared with the fundamental theory, allowing for simple and sometimes striking predictions relating different observables.

1.2. Heavy-Quark Effective Theory

Heavy-quark systems provide prime examples for applications of the EFT technology, because the hierarchy $m_b \gg \Lambda_{\mathrm{QCD}}$ provides a natural separation of scales. Physics at the scale m_b is of a short-distance nature, while for heavy-quark systems there is always also some hadronic physics governed by the confinement scale Λ_{QCD}. Being able to separate the short-distance and long-distance effects associated with these two scales is crucial for any quantitative description in heavy-quark physics. For instance, if the long-distance hadronic matrix elements are obtained from lattice QCD, then it is necessary to analytically compute the short-distance effects, which come from short-wavelength modes that do not fit on present-day lattices. In many other instances, the long-distance hadronic physics can be encoded in a small number of universal parameters. To identify these parameters requires that one first extracts all short-distance effects.

1.2.1. ***General idea and derivation of the effective Lagrangian***: The simplest effective theory for heavy-quark systems is the heavy-quark effective theory (HQET) [5](see [6,7] for a detailed discussion). It provides a simplified description of the soft interactions of a single heavy quark interacting with soft, light partons. This includes the interactions that bind the heavy quark with other light partons inside heavy mesons (B, B^*, ...) and baryons (Λ_b, Σ_b, ...).

A softly interacting heavy quark is nearly on-shell. Its momentum may be decomposed as $p_Q^\mu = m_Q v^\mu + k^\mu$, where v is the 4-velocity of the hadron containing the heavy quark, and the "residual momentum" $k \sim \Lambda_{\mathrm{QCD}}$ results from the soft interactions of the heavy quark with its environment. In the limit $m_Q \gg \Lambda_{\mathrm{QCD}}$, the soft interactions do not change the 4-velocity of the heavy quark, which is therefore a conserved quantum number that is often used as a label on the effective heavy-quark fields. The momentum $m_Q v$ is sometimes referred to as a "label momentum".

A nearly on-shell Dirac spinor has two large and two small components. We define

$$Q(x) = e^{-im_Q v\cdot x}\left[h_v(x) + H_v(x)\right], \tag{1.1}$$

where

$$h_v(x) = e^{im_Q v\cdot x}\,\frac{1+\not v}{2}\,Q(x)\,, \qquad H_v(x) = e^{im_Q v\cdot x}\,\frac{1-\not v}{2}\,Q(x) \tag{1.2}$$

are the large ("upper") and small ("lower") components of the Dirac spinor, respectively. The extraction of the phase factor in Eq. (1.1) implies that the fields h_v and H_v carry the residual momentum k. These fields obey the projection relations $\not v\, h_v = h_v$ and $\not v\, H_v = -H_v$. Inserting these definitions into the Dirac Lagrangian yields

$$\begin{aligned} \mathcal{L}_Q &= \bar h_v\, i\not D\, h_v + \bar H_v\,\left(i\not D - 2m_Q\right)\,H_v + \bar h_v\, i\not D\, H_v + \bar H_v\, i\not D\, h_v \\ &= \bar h_v\, iv\cdot D\, h_v + \bar H_v\,\left(-iv\cdot D - 2m_Q\right)\,H_v + \bar h_v\, i\vec{\not D}\, H_v + \bar H_v\, i\vec{\not D}\, h_v\,, \end{aligned} \tag{1.3}$$

where $i\vec{D}^\mu = iD^\mu - v^\mu\, iv\cdot D$ is the "spatial" covariant derivative (note that $v^\mu = (1,\vec 0)$ in the heavy-hadron rest frame). The interpretation of Eq. (1.3) is that the field h_v describes a massless fermion, while H_v describes a heavy fermion with mass $2m_Q$. Both modes are coupled to each other via the last two terms. Soft interactions cannot excite the heavy fermion, so we integrate it out from the generating functional of the theory. The light field which remains describes the fluctuations of the heavy quark about its mass shell. Solving the classical equation of motion for the field H_v yields

$$H_v = \frac{1}{2m_Q + iv \cdot D}\, i\vec{\not{D}}\, h_v = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(-\frac{iv \cdot D}{2m_Q} \right)^n i\vec{\not{D}}\, h_v \,, \tag{1.4}$$

which implies $H_v = O(\Lambda_{\rm QCD}/m_Q)\, h_v$ provided the residual momenta are small. The leading-order effective Lagrangian obtained from Eq. (1.3) then reads

$$\mathcal{L}_{\rm HQET} = \bar{h}_v\, iv \cdot D_s\, h_v + O\left(1/m_Q\right) \,. \tag{1.5}$$

Note that the covariant derivative $iD_s^\mu = i\partial^\mu + g_s A_s^\mu$ contains only the soft gluon field. Hard gluons have been integrated out. This is the effective Lagrangian of HQET. From it one derives the Feynman rules of the effective theory.

It is straightforward to include power corrections to the effective Lagrangian by keeping higher-order terms in Eq. (1.4). One finds that at subleading order in $1/m_Q$ two new operators arise, such that

$$\mathcal{L}_{\rm HQET} = \bar{h}_v\, iv \cdot D_s\, h_v + \frac{1}{2m_Q} \left[\bar{h}_v \left(i\vec{D}_s \right)^2 h_v + C_{\rm mag}\left(\mu\right) \frac{g_s}{2}\, \bar{h}_v\, \sigma_{\mu\nu}\, G_s^{\mu\nu} h_v \right] + \ldots \,. \tag{1.6}$$

The new operators are referred to as the "kinetic energy" and the "chromo-magnetic interaction". The kinetic-energy operator corresponds to the first correction term in the Taylor expansion of the relativistic energy $E = m_Q + \vec{p}^{\,2}/2m_Q + \ldots$, and Lorentz invariance ensures that its coefficient is not renormalized. The Wilson coefficient of the chromo-magnetic interaction operators has been calculated at NLO in RG-improved perturbation theory [8].

1.2.2. ***Spin-flavor symmetry and applications in spectroscopy***: The leading term in the HQET Lagrangian exhibits a SU($2n_Q$) spin-flavor symmetry. Its physical meaning is that, in the infinite mass limit, the properties of hadronic systems containing a single heavy quark are insensitive to the spin and flavor of the heavy quark [5,9]. The spin symmetry results from the fact that there appear no Dirac matrices in the effective Lagrangian Eq. (1.5), implying that the interactions of the heavy quark with soft gluons leave its spin unchanged. The flavor symmetry arises since the mass of the heavy quark does not appear at leading order. When there are N_h heavy quarks moving at the same velocity, one can simply extend Eq. (1.5) by summing over N_h identical terms for the effective heavy-quark fields h_v^i. The result is then clearly invariant under rotations in flavor space. When combined with the spin symmetry, the symmetry group becomes promoted to SU($2N_h$). The flavor symmetry is broken by the operators arising at order $1/m_Q$ and higher. However, at first order only the chromo-magnetic operator breaks the spin symmetry.

The spin-flavor symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states [10]. In the heavy-quark limit, the spin of the heavy quark and the total angular momentum j of the light degrees of freedom are separately conserved by the strong interactions. Because of heavy quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavor, spin, parity, etc.) of the light degrees of freedom. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm 1/2$. The flavor symmetry relates the properties of states with different heavy quark flavor.

In the case of the ground-state mesons containing a heavy quark, the light degrees of freedom have the quantum numbers of an antiquark, and the degenerate states are the pseudoscalar ($J = 0$) and vector ($J = 1$) mesons. Their masses are split by hyperfine corrections of order $1/m_Q$, such that one expects $m_{B^*} - m_B = O(1/m_b)$ and $m_{D^*} - m_D = O(1/m_c)$. It follows that $m_{B^*}^2 - m_B^2 \simeq m_{D^*}^2 - m_D^2 \simeq$ const. The data are compatible with this result: $m_{B^*}^2 - m_B^2 \simeq 0.49\,\mathrm{GeV}^2$ and $m_{D^*}^2 - m_D^2 \simeq 0.55\,\mathrm{GeV}^2$. One can also study excited meson states, in which the light constituents carry orbital angular momentum. For example, it is tempting to interpret $D_1(2420)$ with $J^P = 1^+$ and $D_2(2460)$ with $J^P = 2^+$ as the spin doublet corresponding to $j = 3/2$. The small mass difference $m_{D_2^*} - m_{D_1} \simeq 35\,\mathrm{MeV}$ supports this assertion. A typical prediction of the flavor symmetry is that the excitation energies for states with different quantum numbers of the light degrees of freedom are approximately the same in the charm and bottom systems. This explains, for example, why the splittings $m_{B_S} - m_B = 89 \pm 5\,\mathrm{MeV}$ and $m_{D_s} - m_D \simeq 100\,\mathrm{MeV}$ are very close to each other.

1.2.3. ***Weak decay form factors***: Of particular interest are the relations between the weak decay form factors of heavy mesons, which parametrize hadronic matrix elements of currents between two meson states containing a heavy quark. These relations have been derived by Isgur and Wise [9], generalizing ideas developed by Nussinov and Wetzel [11] and Voloshin and Shifman [12]. For the purpose of this discussion, it is convenient to work with a mass-independent normalization of meson states. In fact, it is more natural for heavy quark systems to use velocity rather than momentum variables. We will thus write $|M(v)\rangle$ instead of $|M(p)\rangle$.

Consider the elastic scattering of a pseudoscalar meson, $P(v) \to P(v')$, induced by an external vector current coupled to the heavy quark contained in P. Before the action of the current, the light degrees of freedom in the initial state orbit around the heavy quark, which acts as a source of color moving with the meson's velocity v. The action of the current is to replace instantaneously the color source by one moving at velocity v'. If $v = v'$, nothing really happens; the light degrees of freedom do not realize that there was a current acting on the heavy quark. If the velocities are different, however, they suddenly find themselves interacting with a moving color source. Soft gluons have to be exchanged in order to rearrange the light degrees of freedom and build the final state meson moving at velocity v'. This rearrangement leads to a form factor suppression, which reflects the fact that as the velocities become more and more different, the probability for an elastic transition decreases. The important observation is that, in the $m_Q \to \infty$ limit, the form factor can only depend on the Lorentz boost $\gamma = v \cdot v'$ connecting the rest frames of the initial and final-state mesons.

The result of this discussion is that in the effective theory, which provides the appropriate framework to consider the limit $m_Q \to \infty$ with the quark velocities kept fixed, the hadronic matrix element describing the scattering process can be written as

$$\langle P\left(v'\right) | \, \bar{h}_{v'} \gamma^\mu h_v \, | P\left(v\right) \rangle = \xi\left(v \cdot v'\right) \left(v + v'\right)^\mu \ , \tag{1.7}$$

with a form factor $\xi(v \cdot v')$ that does not depend on m_Q. Since the matrix element is invariant under complex conjugation combined with an interchange of v and v', the function $\xi(v \cdot v')$ must be real. That there is no term proportional to $(v - v')^\mu$ can be seen by contracting the matrix element with $(v - v')_\mu$, and using $\not{v} h_v = h_v$ and $\bar{h}_{v'} \not{v}' = \bar{h}_{v'}$. One can now use the flavor symmetry to replace the heavy quark Q in one of the meson states by a heavy quark Q' of a different flavor, thereby turning P into another pseudoscalar meson P'. At the same time, the current becomes a flavor-changing vector current. In the infinite mass limit this is a symmetry transformation, under which the effective Lagrangian is invariant. Hence, the matrix element

$$\langle P'\left(v'\right) | \, \bar{h}'_{v'} \gamma^\mu h_v \, | P\left(v\right) \rangle = \xi\left(v \cdot v'\right) \left(v + v'\right)^\mu \tag{1.8}$$

is still determined by the same function $\xi(v \cdot v')$. This universal form factor is called the Isgur-Wise function [9].

For equal velocities, the vector current $J^\mu = \bar{h}'_v \gamma^\mu h_v = \bar{h}'_v v^\mu h_v$ is conserved in the effective theory, irrespective of the flavor of the heavy quarks. The corresponding conserved charges are the generators of the flavor symmetry. The diagonal generators simply count the number of heavy quarks, whereas the off-diagonal ones replace a heavy quark by another. It follows that the Isgur-Wise function is normalized at the point of equal velocities: $\xi(1) = 1$. This can easily be understood in terms of the physical picture discussed above: When there is no velocity change, the light degrees of freedom see the same color field and are in an identical configuration before and after the action of the current. There is no form factor suppression. Since $E_{\text{recoil}} = m_{P'} \left(v \cdot v' - 1\right)$ is the recoil energy of the daughter meson P' in the rest frame of the parent meson P, the point $v \cdot v' = 1$ is referred to as the zero recoil limit.

The heavy-quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons. They can be written as [9]

$$\begin{aligned} \langle V'\left(v', \epsilon\right) | \, \bar{h}'_{v'} \gamma^\mu \left(1 - \gamma_5\right) \, h_v \, | P\left(v\right) \rangle = {} & i \epsilon^{\mu\nu\alpha\beta} \, \epsilon^*_\nu \, v'_\alpha v_\beta \, \xi\left(v \cdot v'\right) \\ & - \left[\epsilon^{*\mu} \, \left(v \cdot v' + 1\right) - v'^\mu \, \epsilon^* \cdot v\right] \xi\left(v \cdot v'\right) . \end{aligned} \tag{1.9}$$

Once again, the matrix element is completely described in terms of the universal Isgur-Wise form factor. Eq. (1.8) and Eq. (1.9) summarize the relations imposed by heavy-quark symmetry on the weak decay form factors describing the semileptonic decay processes $B \to D \, \ell \, \bar{\nu}$ and $B \to D^* \ell \, \bar{\nu}$. These relations are model-independent consequences of QCD in the limit where $m_b, m_c \gg \Lambda_{\text{QCD}}$. They play a crucial role in the determination of $| \, V_{cb}|$.

1.2.4. *Residual gauge invariance and decoupling transformation*: The effective Lagrangian $\mathcal{L}_{\rm HQET}$ defines an effective theory for soft interactions of heavy quarks. All hard interactions, including the couplings of heavy quarks to hard gluons, are integrated out. As a result, the effective theory no longer has the full gauge invariance of QCD, but only a residual gauge invariance with respect to gauge transformations that preserve the scaling properties of the fields. These are called "soft gauge transformations" and denoted by $U_s(x)$. The transformation rules are

$$\begin{aligned} &h_v\left(x\right) \to U_s\left(x\right)\, h_v\left(x\right), \\ &A_s^\mu\left(x\right) \to U_s\left(x\right)\, A_s^\mu\left(x\right)\, U_s^\dagger\left(x\right) + \frac{i}{g_s}\, U_s\left(x\right)\left[\partial^\mu, U_s^\dagger\left(x\right)\right]. \end{aligned} \tag{1.10}$$

Operationally, "soft" functions like $A_s^\mu(x)$ and $U_s(x)$ can be defined via a restriction to soft modes in their Fourier decomposition. In practice, however, the use of dimensional regularization makes it unnecessary to introduce the hard cutoffs associated with this construction.

The couplings of soft gluons to heavy quarks can be "removed" by the field redefinition $h_v(x) = Y_v(x)\, h_v^{(0)}(x)$, with

$$Y_v\left(x\right) = P \exp\left(ig_s \int_{-\infty}^0 dt\, v\cdot A_s\left(x+tv\right)\right) \tag{1.11}$$

a time-like Wilson line extending from minus infinity to the point x. The symbol P means an ordering with respect to t such that gauge fields are ordered from left to right in the order of decreasing t values. The Wilson line $Y_v^\dagger(x)$ is given by a similar expression but with the opposite ordering prescription, and with ig_s replaced by $-ig_s$ in the exponent. The soft Wilson line obeys the important property $Y_v^\dagger\, iv\cdot D_s\, Y_v = iv\cdot\partial$. Using this relation, it follows that in terms of the new fields the HQET Lagrangian becomes

$$\mathcal{L}_{\rm HQET} = \bar{h}_v^{(0)}\, iv\cdot\partial\, h_v^{(0)} + O\left(1/m_Q\right). \tag{1.12}$$

At leading order in $1/m_Q$, this is a free theory as far as the strong interactions of heavy quarks are concerned. However, the theory is nevertheless non-trivial in the presence of external sources. Consider, e.g., the case of a flavor-changing weak-interaction current, which turns a heavy b-quark into a c-quark (plus a W boson). At tree level, matching such a current onto HQET gives

$$\bar{c}\gamma^\mu\left(1-\gamma_5\right)b \to \bar{h}_{v'}\gamma^\mu\left(1-\gamma_5\right)h_v = \bar{h}_{v'}^{(0)}\gamma^\mu\left(1-\gamma_5\right)\left(Y_{v'}^\dagger Y_v\right)h_v^{(0)}. \tag{1.13}$$

Here v and v' are the velocities of the heavy mesons containing the heavy quarks. Unless the two velocities are equal, the object $Y_{v'}^\dagger Y_v$ is non-trivial, and hence the soft gluons do not decouple from the heavy quarks inside the current operator. Indeed, the gauge-invariant object

$$\xi\left(v\cdot v'\right) = \frac{1}{N_c}\,\mathrm{Tr}\left(Y_{v'}^\dagger Y_v\right) \tag{1.14}$$

is nothing but the Isgur-Wise function introduced in Eq. (1.7). If we close the integration contour at $t = -\infty$, we may interpret $Y_{v'}^{\dagger} Y_v$ as a Wilson loop with a cusp at the point x, where the two paths parallel to the different velocity vectors intersect. The presence of the cusp leads to non-trivial UV behavior (for $v \neq v'$), which is described by a cusp anomalous dimension $\Gamma_c(v \cdot v')$, which was calculated at two-loop order as early as in 1987 [13]. The cusp anomalous dimension is nothing but the celebrated velocity-dependent anomalous dimension of heavy-quark currents, which was rediscovered three years later in the context of HQET [14].

The interpretation of heavy quarks as Wilson lines is very useful, and it was put forward in some of the very first papers on the subject [15]. This technology will be useful in the study of the interactions of heavy quarks with collinear degrees of freedom, which will be discussed in the context of SCET.

1.2.5. ***Model-independent determination of $|V_{cb}|$***: The known normalization of the Isgur-Wise function at zero recoil can be used to obtain a model-independent measurement of the element $|V_{cb}|$ of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The semileptonic decay $B \to D^* \ell \bar{\nu}$ is ideally suited for this purpose [16]. Experimentally, this is a particularly clean mode, since the reconstruction of the D^* meson mass provides a powerful rejection against background. From the theoretical point of view, it is ideal since the decay rate at zero recoil is protected by Luke's theorem against first-order power corrections in $1/m_Q$ [17]. Introducing the recoil variable $w = v \cdot v'$, one finds

$$\lim_{w \to 1} \frac{1}{\sqrt{w^2 - 1}} \frac{d\Gamma\left(\bar{B} \to D^* \ell \bar{\nu}\right)}{dw} = \frac{G_F^2 \,|\, V_{cb}|^2}{4\pi^3} \left(m_B - m_{D^*}\right)^2 m_{D^*}^3 \,|h_{A_1}(1)\,|^2 \,, \qquad (1.15)$$

where the form factor $h_{A_1}(1)$ equals 1 up to calculable short-distance corrections and second-order power corrections $\sim 1/m_Q^2$. In practice, one has to rely on an extrapolation over some range in w to obtain a measurement of $|V_{cb}|$. The shape of the form factor near $w = 1$ is highly constrained by unitarity and analyticity considerations [18,19].

1.2.6. ***Heavy-quark expansion for inclusive decays***: The theoretical description of inclusive decays of hadrons containing a heavy quark exploits two observations [20,21,22,23,24,25,26]: bound-state effects related to the initial state can be calculated using the heavy-quark expansion, and the fact that the final state consists of a sum over many hadronic channels eliminates the sensitivity to the properties of individual final-state hadrons. The second feature rests on the hypothesis of quark-hadron duality, i.e. the assumption that decay rates are calculable in QCD after a smearing procedure has been applied [27]. In semileptonic decays, the integration over the lepton spectrum provides a smearing over the invariant hadronic mass of the final state (global duality). For nonleptonic decays, where the total hadronic mass is fixed, the summation over many hadronic final states provides an averaging (local duality). At present, quark-hadron duality cannot be derived from first principles. The validity of global duality (at energies even lower than those relevant in B decays) has been tested experimentally using high-precision data on semileptonic B decays and on hadronic τ decays.

Using the optical theorem, the inclusive decay width of a hadron H_b containing a b quark can be written in the form

$$\Gamma\left(H_b \to X\right) = \frac{1}{M_{H_b}} \,\mathrm{Im}\, \langle H_b| \,\mathbf{T}\, |H_b\rangle \,, \tag{1.16}$$

where the transition operator $\mathbf{T}$ is given by a correlation function of two effective weak Hamiltonians:

$$\mathbf{T} = i \int \mathrm{d}^4x \, T\{\mathcal{H}_{\text{eff}}\left(x\right), \mathcal{H}_{\text{eff}}\left(0\right)\} \,. \tag{1.17}$$

The effective weak Hamiltonian for b-quark decays consists of dimension-6 four-fermion operators and dipole operators [28]. It follows that the leading contributions to the inclusive decay rate in Eq. (1.16) arise from two-loop diagrams. Because of the large mass of the b quark, the momenta flowing through the internal propagators are large. It is thus possible to construct an operator-product expansion (OPE) for the transition operator, in which it is represented as a series of local operators containing two b-quark fields. The operator with the lowest dimension is $\bar{b}b$. The only gauge-invariant operator with dimension 4 is $\bar{b}\, i\not{D}\, b$; however, the equations of motion imply that this operator can be replaced by $m_b\bar{b}b$. The first operator that is different from $\bar{b}b$ has dimension 5 and contains the gluon field. It arises from diagrams in which a soft gluon is emitted from one of the internal lines of the two-loop diagrams. From dimension 6 on, an increasing number of operators appears. For dimensional reasons, the matrix elements of higher-dimensional operators are suppressed by inverse powers of the b-quark mass. Thus, the total inclusive decay rate of a hadron H_b can be written as [21,22]

$$\Gamma(H_b) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} \left\{ c_3 \,\langle\bar{b}b\rangle + c_5 \,\frac{\langle\bar{b}\, g_s\sigma_{\mu\nu}G^{\mu\nu} b\rangle}{m_b^2} + \sum_n c_6^{(n)} \,\frac{\langle O_6^{(n)}\rangle}{m_b^3} + \ldots \right\}, \tag{1.18}$$

where the prefactor arises from the loop integrations, c_i are calculable coefficient functions, and $\langle O\rangle$ are the (normalized) forward matrix elements between H_b states. These matrix elements can be systematically expanded in powers of $1/m_b$ using the heavy-quark effective theory (HQET) [6,29]. The result is [21,22]

$$\begin{aligned} \langle\bar{b}b\rangle &= 1 - \frac{\mu_\pi^2\left(H_b\right) - \mu_G^2\left(H_b\right)}{2m_b^2} + O\!\left(1/m_b^3\right), \\ \frac{\langle\bar{b}\, g_s\sigma_{\mu\nu}G^{\mu\nu} b\rangle}{m_b^2} &= \frac{2\mu_G^2\left(H_b\right)}{m_b^2} + O\!\left(1/m_b^3\right), \end{aligned} \tag{1.19}$$

where $\mu_\pi^2(H_b)$ and $\mu_G^2(H_b)$ are the matrix elements of the heavy-quark kinetic energy and chromomagnetic interaction inside the hadron H_b, respectively [30]. For the ground-state heavy mesons and baryons, the latter ones can be extracted from spectroscopy, e.g. $\mu_G^2(B) = 3(m_{B^*}^2 - m_B^2)/4 \simeq 0.36\,\mathrm{GeV}^2$ and $\mu_G^2(\Lambda_b) = 0$.

A formula analogous to Eq. (1.18) can be derived for differential distributions in inclusive decay processes, assuming that these distributions are integrated over sufficiently large portions of phase space (or smeared in an appropriate way) to ensure quark-hadron duality. Important examples are the distributions in lepton energy ($d\Gamma/dE_\ell$) or lepton invariant mass ($d\Gamma/dq^2$), as well as moments of the invariant hadronic mass distribution, in the semileptonic processes $\bar{B} \to X_u \,\ell\,\bar{\nu}$ and $\bar{B} \to X_c \,\ell\,\bar{\nu}$, as well as the photon energy spectrum ($d\Gamma/dE_\gamma$) in the radiative process $\bar{B} \to X_s\gamma$. While the latter process is primarily used to test the Standard Model and search for hints of new physics, an analysis of decay distributions in the semileptonic processes can be employed to perform a global fit determining the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$ along with heavy-quark parameters such as the quark masses m_b and m_c as well as the hadronic parameters $\mu_\pi^2(B)$, $\mu_G^2(B)$, etc. These determinations provide some of the most accurate values for these parameters [31,32]. For many of these distributions $1/m_b^3$ corrections [**?**] and even higher-order terms are known.

1.2.7. ***Shape functions and non-local power corrections***: In certain regions of phase space, in which the hadronic final state in an inclusive heavy hadron decay is made up of light energetic partons, the local OPE for inclusive decays described above must be replaced by a more complicated expansion involving hadronic matrix elements of non-local light-ray operators. Prominent examples are the radiative decay $\bar{B} \to X_s\gamma$ for large photon energy E_γ near $m_B/2$ (such that the invariant hadronic mass $m_{X_s}^2 = m_B(m_B - 2E_\gamma)$ of the final state is of order $m_B\Lambda_{\rm QCD}$), and the semileptonic decay $\bar{B} \to X_u \,\ell\,\bar{\nu}$ at large lepton energy or small hadronic invariant mass. In these cases, the differential decay rates at leading order in the heavy-quark expansion can be written in the factorized form $d\Gamma \propto H\,J \otimes S$ [33,34], where the hard function H and the jet function J are calculable in perturbation theory. The characteristic scales for these functions are set by m_b and $(m_b\Lambda_{\rm QCD})^{1/2}$, respectively. The soft function

$$S\,(\omega) = \int \frac{dt}{4\pi}\, e^{-i\omega t}\, \langle \bar{B}(v)\,|\, \bar{h}_v(tn)\, Y_n(tn)\, Y_n^\dagger(0)\, h_v(0)\, |\bar{B}\,(v)\rangle \tag{1.20}$$

is a genuinely non-perturbative object, also called a shape function [25]. Here Y_n are soft Wilson lines along a light-like direction n aligned with the momentum of the hadronic final-state jet. The jet function and the shape function share a common variable $\omega \sim \Lambda_{\rm QCD}$, and the symbol $\otimes$ denotes a convolution in this variable. The leading shape function is process independent and describes radiative as well as semileptonic decays.

In higher orders of the heavy-quark expansion, an increasing number of subleading jet and soft functions is required to describe the decay distributions. These have been analyzed in detail at order $1/m_b$ [35,36,37,38,39,40]. The technology for deriving the corresponding factorization theorems relies on SCET and will be discussed in more detail below. An interesting effect arising for $\bar{B} \to X_s\gamma$ decay is that some of the non-local $1/m_b$ corrections remain even in the calculation of the total decay rate, since due to the hadronic substructure of the photon this process is not really inclusive in QCD. This leads to an irreducible theoretical uncertainty in the calculation of the $\bar{B} \to X_s\gamma$ branching ratio and CP asymmetry, whose magnitudes are difficult to estimate [41].

1.3. Soft-collinear Effective Theory

1.3.1. ***General idea of expansion***: As discussed in the previous section, soft gluons that bind a heavy quark inside a heavy meson can not change the virtuality of that heavy quark by a significant amount. The ratio of $\Lambda_{\rm QCD}/m_Q$ provided the expansion parameter in HQET, which is a small parameter since $m_Q \gg \Lambda_{\rm QCD}$.

This obviously does not work when considering light quarks, unless the energy Q of the quarks is large. In this case, the ratio $\Lambda_{\rm QCD}/Q$ provides a small parameter which can be used to construct an effective theory. In fact, the first attempt to construct the relevant effective theory followed exactly the same steps as HQET, and the resulting theory was called large-energy effective theory [42]. However, this theory lacked one crucial ingredient, which meant that it did not in fact correctly reproduce the long-distance physics of full QCD. The lacking ingredient was that light energetic quarks can not only emit soft gluons, but they can also emit collinear gluons (an energetic gluon in the same direction as the original quark), without changing its virtuality. Thus, to fully reproduce the long distance physics of energetic quarks requires their interactions with both soft and collinear particles. The resulting effective theory is therefore called soft-collinear effective theory (SCET) [43,44,45,46].

To be more precise, consider a quark with energy Q and virtuality $m \ll Q$, moving along the direction $\vec{n}$. It is convenient to parameterize the momentum p_n of this particle in terms of its light-cone components, defined by

$$\left(p_n^-, p_n^+, p_n^\perp\right) = \left(\bar{n}\cdot p_n, n\cdot p_n, p_n^\perp\right), \tag{1.21}$$

where $n^\mu = (1, \vec{n})$ and $\bar{n}^\mu = (1, -\vec{n})$ are light-like 4-vectors, and $n \cdot p_\perp = \bar{n}\cdot p_\perp = 0$. Note that we have added a subscript n on the momentum to identify it as a collinear particle in direction n (more precisely, a particle with energy much larger than its virtuality moving along a direction $\vec{n}$). In terms of these light-cone components, the virtuality satisfies $m^2 = p_n^+ p_n^- + p_n^{\perp 2}$. The individual components of the momentum satisfy

$$\left(p_n^-, p_n^+, p_n^\perp\right) \sim \left(Q, m^2/Q, m\right) \equiv Q\left(1, \lambda^2, \lambda\right), \tag{1.22}$$

where $\lambda = m/Q$ is the expansion parameter of SCET.

If such an energetic particle interacts with a soft particle with momentum scaling as

$$\left(p_s^-, p_s^+, p_s^\perp\right) \sim Q\left(\lambda^2, \lambda^2, \lambda^2\right), \tag{1.23}$$

the resulting momentum $p_c + p_s \sim Q(1, \lambda^2, \lambda)$ remains unchanged. It is also obvious that after interacting with another collinear particle in the same direction with momentum q_n, the resulting momentum $p_n + q_n$ still has the same scaling, and therefore the same virtuality. Thus, in physical situations where we are interested in energetic objects with small virtualities compared to their energy, it is the interactions of collinear and soft degrees of freedom that give rise to the long-distance physics. SCET, which is constructed to reproduce this long-distance dynamics, is therefore an effective theory describing the interactions of collinear and soft particles.

Note that the above power counting has the soft momentum to be of order m^2/Q, where m denotes the mass of a collinear system. If the mass of the collinear system is of order $\Lambda_{\rm QCD}$, as would be the case for a single energetic hadron, this power counting becomes no longer viable, since $\Lambda_{\rm QCD}$ provides a natural cutoff to QCD and the soft momentum cannot be below this scale. To describe such systems requires a modified version of SCET, called $\rm SCET_{II}$ [47], in which the scaling of the soft modes is $Q(\lambda, \lambda, \lambda)$. In this review we will focus only on SCET with scaling discussed before, which is sometimes called $\rm SCET_I$. A discussion of some of the issues that arise in $\rm SCET_{II}$ can be found in [47,48,49,50].

1.3.2. ***Leading-order Lagrangian***: The derivation of the SCET Lagrangian follows similar steps as the derivation of the HQET Lagrangian in Section 1.2.1, but care has to be taken to properly account for the interactions of collinear fields with one another. We begin by deriving the Lagrangian for a theory containing only a single type of collinear degrees of freedom and start from the full QCD Lagragian for a massless fermion

$$\mathcal{L}_n = \bar{q}_n\,(x)\, i\not{D}_n\, q_n\,(x)\,. \tag{1.24}$$

We are interested in the interactions of fermion fields $q_n(x)$ with gluon fields $A_n(x)$, which have collinear momentum in the same light-like direction n. Defining two projection operators

$$P_n = \frac{\not{n}\not{\bar{n}}}{4}\,, \qquad P_{\bar{n}} = \frac{\not{\bar{n}}\not{n}}{4}\,, \tag{1.25}$$

which satisfy $P_n + P_{\bar{n}} = 1$, we can write

$$q_n\,(x) = (P_n + P_{\bar{n}})\, q_n\,(x) \equiv \psi_n\,(x) + \Xi_n\,(x)\,. \tag{1.26}$$

In terms of these two fields, the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_n = \bar{\psi}_n\,(x)\,\frac{\not{\bar{n}}}{2}\, in\cdot D_n\,\psi_n\,(x) + \bar{\Xi}_n\,(x)\,\frac{\not{n}}{2}\, i\bar{n}\cdot D_n\,\Xi_n\,(x) \\ + \bar{\psi}_n\,(x)\, i\not{D}_n^{\perp}\,\Xi_n\,(x) + \bar{\Xi}_n\,(x)\, i\not{D}_n^{\perp}\,\psi_n\,(x)\,.\end{aligned} \tag{1.27}$$

From the power counting we know that $\bar{n}\cdot p_n \gg 1$, such that the field $\Xi_n(x)$ has no pole in its propagator, similar to the field $H_v(x)$ in Eq. (1.3). It can therefore be integrated out using its equations of motion

$$\frac{\not{n}}{2}\, i\bar{n}\cdot D_n\,\Xi_n\,(x) = i\not{D}_n^{\perp}\psi_n\,(x)\,. \tag{1.28}$$

Inserting this back into Eq. (1.27), we find

$$\mathcal{L}_n = \bar{\psi}_n\,(x)\left[in\cdot D_n + i\not{D}_n^{\perp}\frac{1}{i\bar{n}\cdot D_n} i\not{D}_n^{\perp}\right]\frac{\not{\bar{n}}}{2}\,\psi_n\,(x)\,. \tag{1.29}$$

While this Lagrangian leads to the correct Feynman rules of SCET, there is one feature that warrants extra discussion. In contrast to the Lagrangian of HQET given in Eq. (1.5), where the derivative scales like the residual momentum k of the heavy quark, the derivatives in Eq. (1.29) pick up both the large momentum components of order Q and $Q\lambda$, and the residual momentum of order $Q\lambda^2$. One can separate the large and residual momentum components using a procedure similar to the HQET case. Separating the collinear momentum into a "label" and a residual component, $p^\mu = P^\mu + k^\mu$, and performing a phase redefinition on the collinear fields $\psi_n(x) = e^{iP\cdot x}\,\xi_n(x)$, derivatives acting on the fields $\xi_n(x)$ now only pick out the residual momentum

$$\partial^\mu \xi_n(x) = k^\mu \xi_n(x)\,. \tag{1.30}$$

Since the label momentum in SCET is not conserved as in HQET, one defines a label operator$\mathcal{P}^\mu$ acting as $\mathcal{P}^\mu \xi_n(x) = P^\mu \xi_n(x)$ [45], as well as a corresponding covariant label operator $\mathcal{D}_n^\mu = \mathcal{P}^\mu + igA_n(x)$. Using this notation, the Lagrangian of SCET can be written as

$$\mathcal{L}_n = \bar{\xi}_n(x)\left[in\cdot D_n + i\not{\mathcal{D}}_n^\perp \frac{1}{i\bar{n}\cdot\mathcal{D}_n} i\not{\mathcal{D}}_n^\perp\right]\frac{\not{\bar{n}}}{2}\,\xi_n(x)\,. \tag{1.31}$$

While Eq. (1.31) looks very similar to Eq. (1.29), the second term in the Lagrangian only depends on label operators, and not any more on derivatives. This shows that the inverse dependence on $\bar{n}\cdot\mathcal{D}_n$ does not introduce any long-distance non-locality into SCET.

An alternative way to understand the separation between large and small momentum components is to derive the Lagrangian of SCET in position space. In this case no label operators are required to describe interactions in SCET, and the dependence on short-distance effects is contained in non-localities at short distances. For more details on this alternative formulation of SCET, see [51,52,53,54].

The final step to complete the Lagrangian of SCET is to include the interactions of collinear fields with soft fields. These interactions can be included by adding the soft gluons to the covariant derivatives, while preserving the power counting. This leads to the final SCET Lagrangian

$$\mathcal{L}_n = \bar{\xi}_n(x)\left[in\cdot D_n + gn\cdot A_s + i\not{\mathcal{D}}_n^\perp \frac{1}{i\bar{n}\cdot\mathcal{D}_n} i\not{\mathcal{D}}_n^\perp\right]\frac{\not{\bar{n}}}{2}\,\xi_n(x)\,. \tag{1.32}$$

The leading-order Lagrangian describing collinear fields in different light-like directions is simply given by the sum of the Lagrangians for each direction n separately, i.e.

$$\mathcal{L} = \sum_n \mathcal{L}_n \tag{1.33}$$

The soft gluons are the same in each individual Lagrangian. For details on subleading correction in λ to SCET, see [47,51,52,55,56,57].

1.3.3. ***Collinear gauge invariance and Wilson lines***: As in the case of HQET described in Section 1.2.4, SCET only contains residual gauge symmetries. One satisfies collinear scaling

$$\left(\bar{n}\cdot\partial_n, n\cdot\partial_n, \partial_n^{\perp}\right) U_n(x) \sim Q\left(1, \lambda^2, \lambda\right) U_n(x)\,, \tag{1.34}$$

and one soft scaling

$$\left(\bar{n}\cdot\partial_n, n\cdot\partial_n, \partial_n^{\perp}\right) U_s(x) \sim Q\left(\lambda^2, \lambda^2, \lambda^2\right) U_s(x)\,. \tag{1.35}$$

The fact that collinear fields in different directions do not transform under the same gauge transformations implies that each collinear sector, containing particles with large momenta along a certain direction, is separately gauge invariant. This is achieved by the introduction of collinear Wilson lines [45]

$$W_n(x) = P\exp\left[-ig\int_{-\infty}^{0} ds\, \bar{n}\cdot A_n(s\bar{n}+x)\right], \tag{1.36}$$

which transform under collinear gauge transformations according to

$$W_n(x) \to U_n(x)\, W_n(x)\,. \tag{1.37}$$

Thus, the combination

$$\chi_n(x) \equiv W_n^{\dagger}(x)\,\psi_n(x) \tag{1.38}$$

is gauge invariant. Operators in SCET are typically constructed from such gauge-invariant collinear fields.

1.3.4. ***Decoupling of soft gluons***: Soft gluons in SCET couple to collinear quarks only through the term $\bar{\xi}_n\, gn\cdot A_s\,\frac{\not{\bar{n}}}{2}\,\xi_n$ in the effective Lagrangian in Eq. (1.32). This coupling is very similar to the coupling of soft gluons to heavy quarks in HQET, and soft gluons in SCET can be decoupled from collinear fields in a way similar as explained in Section 1.2.4. Written in terms of the redefined fields

$$\psi_n(x) = Y_n(x)\,\psi_n^{(0)}(x)\;, \qquad A_n(x) = Y_n(x)\,A_n^{(0)}(x)\,Y_n^{\dagger}(x)\,, \tag{1.39}$$

the soft gluons decouple from the SCET Lagrangian [46]. This fact greatly facilitates proofs of factorization theorems in SCET.

1.3.5. ***Factorization Theorems***: One of the important applications of SCET is to understand how to factorize cross sections involving energetic particles in different directions into simpler pieces that can either be calculated perturbatively or be determined from data. Factorization theorems have been around for much longer than SCET [58,59,60,61,62,63,64,65], and for reviews on the subject, see [66,67,68]. The effective theory allows for a conceptually simpler understanding of factorization [69,70,71,72,73,74], since most simplifications happen already at the level of the Lagrangian.

As discussed in the previous section, the Lagrangian of SCET does not involve any couplings between collinear degrees of freedom in different light-like directions, or between soft and collinear degrees of freedom after the field redefinition Eq. (1.39) has been performed. An operator describing the scattering and production of collinear partons at short distances can thus be written as

$$\langle O(x)\rangle \simeq C_O(\mu)\left\langle \mathcal{C}^{(0)}_{n_a}(x)\,\mathcal{C}^{(0)}_{n_b}(x)\,\mathcal{C}^{(0)}_{n_1}(x)\ldots\mathcal{C}^{(0)}_{n_N}(x)\left[\mathcal{Y}_{n_a}\mathcal{Y}_{n_b}\mathcal{Y}_{n_1}\ldots\mathcal{Y}_{n_N}\right](x)\right\rangle_\mu . \tag{1.40}$$

Here $\mathcal{C}_n(x)$ denotes a gauge-invariant combination of collinear fields (either quark or gluon fields) in the direction n and the matching coefficient is denoted by C_O. The soft Wilson lines can either be in a color triplet or color octet representation, and are collectively denoted by $\mathcal{Y}_n$. Note that both the matrix elements and the coefficient C_O depend on the renormalization scale μ.

Having defined the operator mediating a given process, one can calculate the cross section by squaring the operator, taking the forward matrix element and integrating over the phase space of all final-state particles,

$$\mathrm{d}\sigma \sim \left\langle \mathrm{in}\left| O(x)\,O^\dagger(0)\right|\mathrm{in}\right\rangle \mathrm{d}\Phi\,, \tag{1.41}$$

where $\mathrm{d}\Phi$ denotes the phase space integration over the final-state particles as well as over the momentum fractions of the incoming partons,

$$\mathrm{d}\Phi \equiv \mathrm{d}x_a\mathrm{d}x_b\prod_i \frac{\mathrm{d}^3p_i}{(2\pi)^3\,2E_i}\delta^4\left(p_{\mathrm{in}}-p_{\mathrm{out}}\right). \tag{1.42}$$

The absence of interactions between collinear degrees of freedom moving along different directions or soft degrees of freedom implies that the forward matrix element can be factorized as

$$\begin{aligned}\left\langle \mathrm{in}\left|O(x)\,O^\dagger(0)\right|\mathrm{in}\right\rangle &= \left\langle \mathrm{in}_a\left|\mathcal{C}_{n_a}(x)\,\mathcal{C}^\dagger_{n_a}(0)\right|\mathrm{in}_a\right\rangle\left\langle \mathrm{in}_b\left|\mathcal{C}_{n_b}(x)\,\mathcal{C}^\dagger_{n_b}(0)\right|\mathrm{in}_b\right\rangle\\ &\quad\times\left\langle 0\left|\mathcal{C}_{n_1}(x)\,\mathcal{C}^\dagger_{n_1}(0)\right|0\right\rangle\cdots\left\langle 0\left|\mathcal{C}_{n_N}(x)\,\mathcal{C}^\dagger_{n_N}(0)\right|0\right\rangle\\ &\quad\times\left\langle 0\left|\left[\mathcal{Y}_{n_a}\ldots\mathcal{Y}_{n_N}\right](x)\left[\mathcal{Y}_{n_a}\ldots\mathcal{Y}_{n_N}\right]^\dagger(0)\right|0\right\rangle .\end{aligned} \tag{1.43}$$

Thus, the matrix element required for the differential cross section has factorized into a product of simpler structures, each of which can be evaluated separately.

The matrix elements of incoming collinear fields are non-perturbative objects given in terms of the well-known parton distribution functions, while the matrix elements of outgoing collinear fields are determined by perturbatively calculable jet functions. Finally, the vacuum matrix element of the soft Wilson lines defines a so-called soft function. The common dependence on x implies that in momentum space the various components of the factorization theorem are convoluted with one another. Deriving this convolution requires a careful treatment of the phase-space integration, in particular treating the large and residual components of each momentum appropriately. While this is a straightforward exercise, the details will be omitted here (see [52,74]) .

Putting all information together, the differential cross section can be written as

$$\mathrm{d}\sigma \sim H(\mu) \otimes \left[f_{p_1/P}(\mu)\, f_{p_2/P}(\mu)\right] \otimes \left[J_1(\mu) \ldots J_N(\mu)\right] \otimes S(\mu)\,. \tag{1.44}$$

Here the hard coefficient $H(\mu)$ is equal to the square of the matching coefficient $H(\mu) = |C_O(\mu)|^2$. It should be mentioned that the most difficult part of traditional factorization proofs involves showing that so-called Glauber gluons do not spoil the above factorization theorem. So far, Glauber gluons are not included in the SCET derivations of factorization theorems, but preliminary work on understanding Glauber gluons in SCET exists [75].

1.3.6. ***Resummation of large logarithms***: SCET can be used to sum the large logarithmic terms that arise in perturbative calculations. In general, perturbation theory will generate a logarithmic dependence on any ratio of scales in a problem, and for processes that involve initial or final states with energy much in excess of their mass there are two powers of logarithms for every power of the strong coupling constant. Thus, for widely separated scales these large logarithms can spoil fixed-order perturbation theory, and a much better convergence is achieved by expanding in α_s, but holding $\alpha_s \log^2(r)$ fixed, such the first term in the new expansion resumms powers of $\alpha_s \log^2(r)$ to all orders. To be more precise, a proper resumption requires to sum logarithms of the form $\alpha_s^n \log^m$ in the exponent in the logarithm of a cross-section.

The important ingredient in achieving this resumption is the fact that SCET factorizes a given cross section into simpler pieces, as discussed in the previous section. Each of the ingredients of the factorization theorem depends on a single physical scale, and the only dependence on that scale can arise through logarithms of its ratio with the renormalization scale μ. Thus, for each of the components in the factorization theorem one can choose a renormalization scale μ for which the large logarithmic terms are absent.

Of course, the factorization formula requires a common renormalization scale μ in all its components, and one therefore has to use the renormalization group (RG) to evolve the various component functions from their preferred scale to the common scale μ. For example, for the hard coefficient $H(\mu)$, the RG equation can be written as

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} H(\mu) = \gamma_H(Q,\mu)\, H(\mu)\,. \tag{1.45}$$

In general, the anomalous dimension is of the form $\gamma_H(\mu) = c_H\, \Gamma_{\rm cusp}(\alpha_s) \log(Q/\mu) + \gamma(\alpha_s)$, where c_H is a process-dependent coefficient and $\Gamma_{\rm cusp}$ denotes the so-called cusp anomalous dimension [13,76]. The non-cusp part of the anomalous dimension γ is again process dependent. The presence of a logarithm of the hard scale Q in the anomalous dimension is characteristic of Sudakov problems and arises since the perturbative series contains double logarithms of scale ratios. The anomalous dimension γ_H is known at two-loop order for arbitrary n-parton amplitudes containing massless or massive external partons [77,78,79,80]. Solving the RG equation yields

$$H\left(\mu\right) = U_H\left(\mu,\mu_h\right) H\left(\mu_h\right) \,, \tag{1.46}$$

which can be used to write the hard function at a scale $\mu_h \sim Q$, where its perturbative expression does not contain any large logarithms, in terms of the common renormalization scale μ. The RG evolution factor $U_H(\mu,\mu_h)$ sums logarithms of the form μ/μ_h. By calculating the anomalous dimension $\gamma_H(\mu)$ to higher and higher orders in perturbation theory, one can resum more and more logarithms in the evolution kernel. The RG equations for the jet and soft functions (as well as for the parton distribution functions) are more complicated, since they involve convolutions over the relevant momentum variables. This will not be discussed in detail here, but for more details, see [44,81,82,83].

1.3.7. ***Applications***: SCET has many applications. Most of these are either in flavor physics, where the decay of a heavy B meson can give rise to energetic light partons, or in collider physics, where the presence of jets naturally leads to collimated sets of energetic particles.

As already mentioned in Section 1.3.1, applications in flavor physics which involve exclusive B-meson decays into light, energetic mesons require $\mathrm{SCET_{II}}$ [84,85,86], which is beyond the scope of this review. However, SCET has been successfully applied to the decays $B \to X_s\gamma$ and $B \to X_u\ell\nu$ decay, as was already discussed in Section 1.2.7. Following the steps outlined in Section 1.3.5, SCET factorization theorems have been proven, which for both decays take the symbolic form

$$\mathrm{d}\sigma \sim H\left(\mu\right) J\left(\mu\right) \otimes S\left(\mu\right). \tag{1.47}$$

While the hard function is different for the two decays, the jet and soft functions are identical at leading order in $\Lambda_{\mathrm{QCD}}/m_Q$. This is particularly important for the soft function, given by the matrix element in Eq. (1.20) (where Y_n Wilson lines in HQET are identical to the Y_n Wilson lines in SCET). It is this shape function that introduces non-perturbative physics into the theoretical predictions for the cross sections of $B \to X_s\gamma$ and $B \to X_u\ell\nu$ in the regions of experimental interest. The fact that both decays depend on the same non-perturbative function has allowed to determine this non-perturbative information from the measured shape of the photon spectrum in $B \to X_s\gamma$, allowing for a better understanding of the process used to determine the CKM element $|V_{ub}|$.

Another important application of SCET is its use to obtain precision calculations for event-shape distributions at e^+e^- colliders. Such observables, in particular the thrust distribution, have been measured to high accuracy at LEP (for a review, see [87]) . Comparing these data to precise theoretical predictions allows for a determination of the strong coupling constant α_s. For small values of thrust ($\tau \ll 1$), the distribution can be factorized into the form [88,89,90]

$$\frac{1}{Q\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = H(\mu) \int \mathrm{d}s \int \mathrm{d}k\, J(s,\mu)\; S(Q\tau - s/Q - k, \mu)\,. \tag{1.48}$$

Here Q denotes the center-of-mass energy of the collision, σ_0 is the total hadronic cross section, and H, J and S are the hard, jet and soft functions in SCET. Large logarithms of the form $(\alpha_s^n \ln^{2n-1}\tau)/\tau$ become important and have to be resummed. Furthermore, for $\tau \sim \Lambda_{\mathrm{QCD}}/Q$ non-perturbative effects in the soft function become important. SCET allows to include both the resummation of large logarithmic terms as well as the non-perturbative physics through a shape function [91], very similar to the B-physics case discussed above. The known perturbative effects for large values of τ can be included by matching the SCET result to the known two-loop spectrum. SCET has also been applied to study more complicated event shapes, for which all-order factorization theorems were not available before. An example is the jet broadening, for which the factorization theorem is affected by the so-called collinear anomaly [92].

A final application worth mentioning is that of soft-gluon (or threshold) resummation at hadron colliders. When producing a given final state close to threshold, extra radiation is restricted to be soft, since there is not enough energy available for hard emissions. This gives rise to large double-logarithmic corrections in the perturbative expansion, which can be resummed. A prominent example is the Drell-Yan process, when the invariant mass m of the lepton pair is approaching the partonic center-of-mass energy $\sqrt{\hat{s}}$. In this case, the large logarithms are of the form $\log(1 - m^2/\hat{s})$, and their resummation was studied in [93,94]. The resummation of these logarithms in SET using the techniques mentioned in Section 1.3.6 has been performed in [95]. The same approach has also been applied to closely related processes such as Higgs and top-quark pair production [99,100]. The fact that at a hadron collider one needs to integrate over all possible values of the partonic center-of-mass energy implies that, for a given value of m, only in a small portion of phase space the threshold logarithms are large. However, it has been argued that threshold resummation is nevertheless phenomenologically important, because the partonic threshold region is strongly enhanced due to the steepness of the parton luminosities [95,96,97,98].

1.3.8. *Open issues and perspectives*: SCET is still a rapidly developing field, and there are several open questions that need to be answered. In this review we have not discussed any issues having to do with $\mathrm{SCET_{II}}$, which is the appropriate effective theory describing interactions of collinear particles interacting with soft particles having momentum scaling as $Q(\lambda,\lambda,\lambda)$. This is important, for example to describe decays of B mesons to light, energetic mesons, or in collider applications such as p_T resummations. There are still many open issues in how to properly formulate $\mathrm{SCET_{II}}$, which are under active investigation. They include the treatment of endpoint singularities of convolution integrals, double counting between overlapping momentum regions, and the breakdown of the naive factorization of soft and collinear modes due to quantum effects. In the review we have mentioned the issue of Glauber gluons in passing. Glauber gluons are known to affect factorization theorems, but how to properly include them in SCET is still an open question.

1. E. Witten, Nucl. Phys. B **122**, 109 (1977).
2. S. Weinberg, Phys. Lett. B **91**, 51 (1980).
3. L. J. Hall, Nucl. Phys. B **178**, 75 (1981).
4. J. Polchinski, hep-th/9210046.
5. H. Georgi, Phys. Lett. B **240**, 447 (1990).
6. M. Neubert, Phys. Rept. **245**, 259 (1994) [hep-ph/9306320].
7. A. V. Manohar and M. B. Wise, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **10**, 1 (2000).
8. G. Amoros, M. Beneke and M. Neubert, Phys. Lett. B **401**, 81 (1997) [hep-ph/9701375].
9. N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); N. Isgur and M. B. Wise, Phys. Lett. B **237**, 527 (1990).
10. N. Isgur and M. B. Wise, Phys. Rev. Lett. **66**, 1130 (1991).
11. S. Nussinov and W. Wetzel, Phys. Rev. D **36**, 130 (1987).
12. M. A. Shifman and M. B. Voloshin, Sov. J. Nucl. Phys. **45**, 292 (1987); Sov. J. Nucl. Phys. **47**, 511 (1988).
13. G. P. Korchemsky and A. V. Radyushkin, Nucl. Phys. B **283**, 342 (1987).
14. A. F. Falk, H. Georgi, B. Grinstein and M. B. Wise, Nucl. Phys. B **343**, 1 (1990).
15. E. Eichten and B. Hill, Phys. Lett. B **234**, 511 (1990); Phys. Lett. B **243**, 427 (1990).
16. M. Neubert, Phys. Lett. B **264**, 455 (1991).
17. M. E. Luke, Phys. Lett. B **252**, 447 (1990).
18. C. G. Boyd, B. Grinstein and R. F. Lebed, Phys. Rev. D **56**, 6895 (1997) [hep-ph/9705252].
19. I. Caprini, L. Lellouch and M. Neubert, Nucl. Phys. B **530**, 153 (1998) [hep-ph/9712417].
20. J. Chay, H. Georgi and B. Grinstein, Phys. Lett. B **247**, 399 (1990).
21. I. I. Y. Bigi, N. G. Uraltsev and A. I. Vainshtein, Phys. Lett. B **293**, 430 (1992) [hep-ph/9207214]; I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, Phys. Rev. Lett. **71**, 496 (1993) [hep-ph/9304225]; B. Blok, L. Koyrakh, M. A. Shifman and A. I. Vainshtein, Phys. Rev. D **49**, 3356 (1994) [hep-ph/9307247].
22. A. V. Manohar and M. B. Wise, Phys. Rev. D **49**, 1310 (1994) [hep-ph/9308246].
23. A. F. Falk, M. E. Luke and M. J. Savage, Phys. Rev. D **49**, 3367 (1994) [hep-ph/9308288].
24. A. F. Falk, Z. Ligeti, M. Neubert and Y. Nir, Phys. Lett. B **326**, 145 (1994) [hep-ph/9401226].
25. M. Neubert, Phys. Rev. D **49**, 3392 (1994) [hep-ph/9311325]; Phys. Rev. D **49**, 4623 (1994) [hep-ph/9312311].

26. I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, Int. J. Mod. Phys. A **9**, 2467 (1994) [hep-ph/9312359].
27. E. C. Poggio, H. R. Quinn and S. Weinberg, Phys. Rev. D **13**, 1958 (1976).
28. For a review, see: G. Buchalla, A. J. Buras and M. E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996) [hep-ph/9512380].
29. M. Neubert, Int. J. Mod. Phys. A **11**, 4173 (1996) [hep-ph/9604412].
30. A. F. Falk and M. Neubert, Phys. Rev. D **47**, 2965 (1993) [hep-ph/9209268].
31. C. W. Bauer, Z. Ligeti, M. Luke and A. V. Manohar, Phys. Rev. D **67**, 054012 (2003) [hep-ph/0210027].
32. C. W. Bauer, Z. Ligeti, M. Luke, A. V. Manohar and M. Trott, Phys. Rev. D **70**, 094017 (2004) [hep-ph/0408002].
33. G. P. Korchemsky and G. F. Sterman, Phys. Lett. B **340**, 96 (1994) [hep-ph/9407344].
34. S. W. Bosch, B. O. Lange, M. Neubert and G. Paz, Nucl. Phys. B **699**, 335 (2004) [hep-ph/0402094].
35. C. W. Bauer, M. E. Luke and T. Mannel, Phys. Rev. **D68**, 094001 (2003) [hep-ph/0102089].
36. C. W. Bauer, M. Luke and T. Mannel, Phys. Lett. B **543**, 261 (2002) [hep-ph/0205150].
37. A. K. Leibovich, Z. Ligeti and M. B. Wise, Phys. Lett. **B539**, 242-248 (2002). [hep-ph/0205148].
38. K. S. M. Lee and I. W. Stewart, Nucl. Phys. B **721**, 325 (2005) [hep-ph/0409045].
39. S. W. Bosch, M. Neubert and G. Paz, JHEP **0411**, 073 (2004) [hep-ph/0409115].
40. M. Beneke, F. Campanario, T. Mannel and B. D. Pecjak, JHEP **0506**, 071 (2005) [hep-ph/0411395].
41. M. Benzke, S. J. Lee, M. Neubert and G. Paz, JHEP **1008**, 099 (2010) [arXiv:1003.5012 [hep-ph]]; Phys. Rev. Lett. **106**, 141801 (2011) [arXiv:1012.3167 [hep-ph]].
42. M. J. Dugan and B. Grinstein, Phys. Lett. B **255**, 583 (1991).
43. C. W. Bauer, S. Fleming and M. E. Luke, Phys. Rev. D **63**, 014006 (2000) [hep-ph/0005275].
44. C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, Phys. Rev. D **63**, 114020 (2001) [hep-ph/0011336].
45. C. W. Bauer and I. W. Stewart, Phys. Lett. B **516**, 134 (2001) [hep-ph/0107001].
46. C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **65**, 054022 (2002) [hep-ph/0109045].
47. C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **67**, 071502 (2003) [hep-ph/0211069].
48. T. Becher, R. J. Hill and M. Neubert, Phys. Rev. D **69**, 054017 (2004) [hep-ph/0308122].

49. T. Becher, R. J. Hill, B. O. Lange and M. Neubert, Phys. Rev. D **69**, 034013 (2004) [hep-ph/0309227].
50. C. W. Bauer, M. P. Dorsten and M. P. Salem, Phys. Rev. D **69**, 114011 (2004) [hep-ph/0312302].
51. J. Chay and C. Kim, Phys. Rev. D **65**, 114016 (2002) [hep-ph/0201197].
52. M. Beneke, A. P. Chapovsky, M. Diehl and T. Feldmann, Nucl. Phys. B **643**, 431 (2002) [hep-ph/0206152].
53. R. J. Hill and M. Neubert, Nucl. Phys. B **657**, 229 (2003) [hep-ph/0211018].
54. M. Beneke and T. Feldmann, Phys. Lett. B **553**, 267 (2003) [hep-ph/0211358].
55. C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **66**, 054005 (2002) [hep-ph/0205289].
56. D. Pirjol and I. W. Stewart, Phys. Rev. D **67**, 094005 (2003) [Erratum-ibid. D **69**, 019903 (2004)] [hep-ph/0211251].
57. C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **68**, 034021 (2003) [hep-ph/0303156].
58. R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer and G. G. Ross, Phys. Lett. B **78**, 281 (1978).
59. H. D. Politzer, Nucl. Phys. B **129**, 301 (1977).
60. C. T. Sachrajda, Phys. Lett. B **73**, 185 (1978).
61. G. T. Bodwin, S. J. Brodsky and G. P. Lepage, Phys. Rev. Lett. **47**, 1799 (1981).
62. G. T. Bodwin, Phys. Rev. D **31**, 2616 (1985) [Erratum-ibid. D **34**, 3932 (1986)] [Phys. Rev. D **34**, 3932 (1986)].
63. J. C. Collins, D. E. Soper and G. F. Sterman, Phys. Lett. B **109**, 388 (1982).
64. J. C. Collins, D. E. Soper and G. F. Sterman, Nucl. Phys. B **263**, 37 (1986).
65. J. C. Collins, D. E. Soper and G. F. Sterman, Nucl. Phys. B **308**, 833 (1988).
66. J. C. Collins and D. E. Soper, Ann. Rev. Nucl. Part. Sci. **37**, 383 (1987).
67. G. F. Sterman, hep-ph/9606312.
68. R. L. Jaffe, hep-ph/9602236.
69. C. W. Bauer, A. V. Manohar and M. B. Wise, Phys. Rev. Lett. **91**, 122001 (2003) [hep-ph/0212255].
70. C. W. Bauer, S. Fleming, D. Pirjol, I. Z. Rothstein and I. W. Stewart, Phys. Rev. D **66**, 014017 (2002) [hep-ph/0202088].
71. C. W. Bauer, C. Lee, A. V. Manohar and M. B. Wise, Phys. Rev. D **70**, 034014 (2004) [hep-ph/0309278].
72. C. Lee and G. F. Sterman, Phys. Rev. D **75**, 014022 (2007) [hep-ph/0611061].
73. C. W. Bauer, S. P. Fleming, C. Lee and G. F. Sterman, Phys. Rev. D **78**, 034027 (2008) [arXiv:0801.4569 [hep-ph]].
74. C. W. Bauer, A. Hornig and F. J. Tackmann, Phys. Rev. D **79**, 114013 (2009) [arXiv:0808.2191 [hep-ph]].

75. C. W. Bauer, B. O. Lange and G. Ovanesyan, JHEP **1107**, 077 (2011) [arXiv:1010.1027 [hep-ph]].
76. I. A. Korchemskaya and G. P. Korchemsky, Phys. Lett. B **287**, 169 (1992)..
77. T. Becher and M. Neubert, Phys. Rev. Lett. **102**, 162001 (2009) [arXiv:0901.0722 [hep-ph]]; JHEP **0906**, 081 (2009) [arXiv:0903.1126 [hep-ph]]; T. Becher, M. Neubert, Phys. Rev. D **79**, 125004 (2009) [arXiv:0904.1021 [hep-ph]].
78. E. Gardi and L. Magnea, JHEP **0903**, 079 (2009) [arXiv:0901.1091 [hep-ph]].
79. A. Mitov, G. F. Sterman, I. Sung, Phys. Rev. D **79**, 094015 (2009) [arXiv:0903.3241 [hep-ph]].
80. A. Ferroglia, M. Neubert, B. D. Pecjak and L. L. Yang, Phys. Rev. Lett. **103**, 201601 (2009) [arXiv:0907.4791 [hep-ph]]; A. Ferroglia, M. Neubert, B. D. Pecjak and L. L. Yang, JHEP **0911**, 062 (2009) [arXiv:0908.3676 [hep-ph]].
81. M. Neubert, Phys. Rev. D **72**, 074025 (2005) [hep-ph/0506245].
82. T. Becher and M. Neubert, Phys. Rev. Lett. **97**, 082001 (2006) [hep-ph/0605050].
83. S. Fleming, A. H. Hoang, S. Mantry and I. W. Stewart, Phys. Rev. D **77**, 114003 (2008) [arXiv:0711.2079 [hep-ph]].
84. C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. Lett. **87**, 201806 (2001) [hep-ph/0107002].
85. C. W. Bauer, D. Pirjol, I. Z. Rothstein and I. W. Stewart, Phys. Rev. D **70**, 054015 (2004) [hep-ph/0401188].
86. T. Becher, R. J. Hill and M. Neubert, Phys. Rev. D **72**, 094017 (2005) [hep-ph/0503263].
87. S. Kluth, Rept. Prog. Phys. **69**, 1771 (2006) [hep-ex/0603011].
88. C. F. Berger and G. F. Sterman, JHEP **0309**, 058 (2003) [hep-ph/0307394].
89. C. F. Berger and G. F. Sterman, Eur. Phys. J. C **33**, S407 (2004) [hep-ph/0310058].
90. M. D. Schwartz, Phys. Rev. **D77**, 014026 (2008) [arXiv:0709.2709 [hep-ph]].
91. R. Abbate, M. Fickinger, A. H. Hoang, V. Mateu, I. W. Stewart, Phys. Rev. **D83**, 074021 (2011) [arXiv:1006.3080 [hep-ph]].
92. T. Becher, G. Bell and M. Neubert, arXiv:1104.4108 [hep-ph], to appear in Phys. Lett. B.
93. G. F. Sterman, Nucl. Phys. B **281**, 310 (1987).
94. S. Catani and L. Trentadue, Nucl. Phys. B **327**, 323 (1989).
95. T. Becher, M. Neubert and G. Xu, JHEP **0807**, 030 (2008) [arXiv:0710.0680 [hep-ph]].
96. D. Appell, G. F. Sterman and P. B. Mackenzie, Nucl. Phys. B **309**, 259 (1988).
97. S. Catani, M. L. Mangano and P. Nason, JHEP **9807**, 024 (1998) [hep-ph/9806484].
98. C. W. Bauer, N. D. Dunn and A. Hornig, arXiv:1010.0243 [hep-ph].
99. V. Ahrens, T. Becher, M. Neubert and L. L. Yang, Eur. Phys. J. C **62**, 333 (2009) [arXiv:0809.4283 [hep-ph]].

100. V. Ahrens, A. Ferroglia, M. Neubert, B. D. Pecjak and L. L. Yang, JHEP **1009**, 097 (2010) [arXiv:1003.5827 [hep-ph]].